

Quotient groups of IA-automorphisms of a free group of rank 3

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Abstract

We prove that, for any positive integer c , the quotient group $\gamma_c(M_3)/\gamma_{c+1}(M_3)$ of the lower central series of the McCool group M_3 is isomorphic to two copies of the quotient group $\gamma_c(F_3)/\gamma_{c+1}(F_3)$ of the lower central series of a free group F_3 of rank 3 as \mathbb{Z} -modules. Furthermore, we give a necessary and sufficient condition whether the associated graded Lie algebra $\text{gr}(M_3)$ of M_3 is naturally embedded into the Johnson Lie algebra $\mathcal{L}(\text{IA}(F_3))$ of the IA-automorphisms of F_3 .

1 Introduction and Notation

Let G be a group. For a positive integer c , let $\gamma_c(G)$ be the c -th term of the lower central series of G . We point out that $\gamma_2(G) = G'$; that is, the derived group of G . We write $\text{IA}(G)$ for the kernel of the natural group homomorphism from $\text{Aut}(G)$ into $\text{Aut}(G/G')$ and we call its elements *IA-automorphisms* of G . For a positive integer $c \geq 2$, the natural group epimorphism from G onto $G/\gamma_c(G)$ induces a group homomorphism π_c from $\text{Aut}(G)$ of G into $\text{Aut}(G/\gamma_c(G))$ of $G/\gamma_c(G)$. Write $\text{I}_c\text{A}(G) = \text{Ker}\pi_c$. Note that $\text{I}_2\text{A}(G) = \text{IA}(G)$. It is proved by Andreadakis [1, Theorem 1.2], that if G is residually nilpotent (that is, $\bigcap_{c \geq 1} \gamma_c(G) = \{1\}$), then $\bigcap_{c \geq 2} \text{I}_c\text{A}(G) = \{1\}$.

Throughout this paper, by “Lie algebra”, we mean Lie algebra over the ring of integers \mathbb{Z} . Let G be a group. Write $\text{gr}_c(G) = \gamma_c(G)/\gamma_{c+1}(G)$ for $c \geq 1$ and denote by (a, b) the commutator $(a, b) = a^{-1}b^{-1}ab$ with $a, b \in G$. The (restricted) direct sum of the quotients $\text{gr}_c(G)$ is the *associated graded Lie algebra* of G , $\text{gr}(G) = \bigoplus_{c \geq 1} \text{gr}_c(G)$. The Lie bracket multiplication in $\text{L}(G)$ is $[a\gamma_{c+1}(G), b\gamma_{d+1}(G)] = (a, b)\gamma_{c+d+1}(G)$, where $a\gamma_{c+1}(G)$ and $b\gamma_{d+1}(G)$ are the images of the elements $a \in \gamma_c(G)$ and $b \in \gamma_d(G)$ in the quotient groups $\text{gr}_c(G)$ and $\text{gr}_d(G)$, respectively, and $(a, b)\gamma_{c+d+1}(G)$ is the image of the group commutator (a, b) in the quotient group $\text{gr}_{c+d}(G)$. Multiplication is then extended to $\text{gr}(G)$ by linearity.

For a positive integer n , with $n \geq 2$, we write F_n for a free group of rank n with a free generating set $\{x_1, \dots, x_n\}$. For $c \geq 2$, we write $F_{n,c-1} = F_n/\gamma_c(F_n)$. Thus, $F_{n,c-1}$ is the free nilpotent group of rank n and class $c-1$. The natural group epimorphism from F_n onto $F_{n,c-1}$ induces a group homomorphism $\pi_{n,c-1} : \text{Aut}(F_n) \rightarrow \text{Aut}(F_{n,c-1})$. We write $\text{I}_c\text{A}(F_n) = \text{Ker}\pi_{n,c-1}$. It is well known that, for $t, s \geq 2$, $(\text{I}_t\text{A}(F_n), \text{I}_s\text{A}(F_n)) \subseteq \text{I}_{t+s-1}\text{A}(F_n)$. Since F_n is residually nilpotent, we have $\bigcap_{c \geq 2} \text{I}_c\text{A}(F_n) = \{1\}$. Since $F_{n,c}/\text{gr}_c(F_n) \cong F_{n,c-1}$ and $\text{gr}_c(F_n)$ is a fully invariant subgroup of $F_{n,c}$, the natural group epimorphism from $F_{n,c}$ onto $F_{n,c-1}$ induces a group homomorphism $\psi_{c,c-1} : \text{Aut}(F_{n,c}) \rightarrow \text{Aut}(F_{n,c-1})$. It is well known that $\psi_{c,c-1}$ is onto. For $c \geq 2$, we define $A_c^*(F_n) = \text{Im}\pi_{n,c} \cap \text{Ker}\psi_{c,c-1}$. For $t \in \{2, \dots, c\}$, the natural group epimorphism from $F_{n,c}$ onto $F_{n,c}/\gamma_t(F_{n,c})$ induces a group homomorphism $\sigma_{c,t} : \text{Aut}(F_{n,c}) \rightarrow \text{Aut}(F_{n,c}/\gamma_t(F_{n,c}))$. Write $\text{I}_t\text{A}(F_{n,c}) = \text{Ker}\sigma_{c,t}$, and, for $t = 2$, $\text{IA}(F_{n,c}) = \text{I}_2\text{A}(F_{n,c})$. We note that, for $c \geq 2$, $F_{n,c}/\gamma_c(F_{n,c}) \cong F_{n,c-1}$. Thus, for $c \geq 2$,

$A_c^*(F_n) = \text{Im} \pi_{n,c} \cap I_c A(F_{n,c})$. It is easily proved that $A_c^*(F_n) \cong I_c A(F_n)/I_{c+1} A(F_n)$ as free abelian groups (see, also, [1, Section 4, p. 246]). Furthermore, for $n, c \geq 2$, $\text{rank}(A_c^*(F_n)) \leq n \text{rank}(\text{gr}_c(F_n)) = \frac{n}{c} \sum_{d|c} \mu(d) n^{c/d}$, where μ is the Möbius function. We point out that $\text{rank}(\text{gr}_c(F_n)) = \frac{1}{c} \sum_{d|c} \mu(d) n^{c/d}$ for all $n, c \geq 2$ (see, for example, [8]).

From now on, for $r \geq 2$, we write $\mathcal{L}^r(\text{IA}(F_n)) = I_r A(F_n)/I_{r+1} A(F_n)$. Form the (restricted) direct sum of the free abelian groups $\mathcal{L}^r(\text{IA}(F_n))$, and denoted by

$$\mathcal{L}(\text{IA}(F_n)) = \bigoplus_{r \geq 2} \mathcal{L}^r(\text{IA}(F_n)).$$

It has the structure of a graded Lie algebra with $\mathcal{L}^r(\text{IA}(F_n))$ as component of degree $r - 1$ in the grading and Lie multiplication given by

$$[\phi I_{j+1} A(F_n), \psi I_{\kappa+1} A(F_n)] = (\phi^{-1} \psi^{-1} \phi \psi) I_{j+\kappa} A(F_n),$$

for all $\phi \in I_j A(F_n)$, $\psi \in I_\kappa A(F_n)$ ($j, \kappa \geq 2$). Multiplication is then extended to $\mathcal{L}(\text{IA}(F_n))$ by linearity. The above Lie algebra is usually called *the Johnson Lie algebra of $\text{IA}(F_n)$* . We point out that, for a positive integer c , $\gamma_c(\text{IA}(F_n)) \subseteq I_{c+1} A(F_n)$.

Let H be a finitely generated subgroup of $\text{IA}(F_n)$ with H/H' torsion-free. For a positive integer q , let $\mathcal{L}_1^q(H) = \gamma_q(H)(I_{q+2} A(F_n))/I_{q+2} A(F_n)$. Form the (restricted) direct sum of abelian groups $\mathcal{L}_1(H) = \bigoplus_{q \geq 1} \mathcal{L}_1^q(H)$. It is easily verified that $\mathcal{L}_1(H)$ is a Lie subalgebra of $\mathcal{L}(\text{IA}(F_n))$. Furthermore, if $\{y_1 H', \dots, y_m H'\}$ is a \mathbb{Z} -basis for H/H' , then $\mathcal{L}_1(H)$ is generated as Lie algebra by the set $\{y_1(I_3 A(F_n)), \dots, y_m(I_3 A(F_n))\}$. By a *natural embedding* of $\text{gr}(H)$ into $\mathcal{L}(\text{IA}(F_n))$, we mean that there exists a Lie algebra isomorphism ϕ from $\text{gr}(H)$ onto $\mathcal{L}_1(H)$ satisfying the conditions $\phi(y_i H') = y_i(I_3 A(F_n))$, $i = 1, \dots, m$. In this case, we also say that $\text{gr}(H)$ is naturally isomorphic to $\mathcal{L}_1(H)$.

For $n \geq 2$, it was shown by Magnus [7], using work of Nielsen [12], that $\text{IA}(F_n)$ has a finite generating set $\{\chi_{ij}, \chi_{ijk} : 1 \leq i, j, k \leq n; i \neq j, k; j < k\}$, where χ_{ij} maps $x_i \mapsto x_i(x_i, x_j)$ and χ_{ijk} maps $x_i \mapsto x_i(x_j^{-1}, x_k^{-1})$, with both χ_{ij} and χ_{ijk} fixing the remaining basis elements. Let M_n be the subgroup of $\text{IA}(F_n)$ generated by the subset $S = \{\chi_{ij} : 1 \leq i, j \leq n; i \neq j\}$. Then, M_n is called the *McCool group* or the *basis conjugating automorphisms group*. It is easily verified that the following relations are satisfied by the elements of S , provided that, in each case, the subscripts i, j, k, q occurring are distinct: $(\chi_{ij}, \chi_{kj}) = (\chi_{ij}, \chi_{kq}) = (\chi_{ij} \chi_{kj}, \chi_{ik}) = 1$. It has been proved in [9] that M_n has a presentation $\langle S \mid Z \rangle$, where Z is the set of all possible relations of the above forms. Since $\gamma_c(M_n) \subseteq \gamma_c(\text{IA}(F_n)) \subseteq I_{c+1} A(F_n)$ for all $c \geq 1$, and since F_n is residually nilpotent, we have $\bigcap_{c \geq 1} \gamma_c(M_n) = \{1\}$ and so, M_n is residually nilpotent.

In the present paper, we show the following result.

Theorem 1 1. For a positive integer c ,

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong \gamma_c(F_3)/\gamma_{c+1}(F_3) \oplus \gamma_c(F_3)/\gamma_{c+1}(F_3)$$

as free abelian groups.

2. Let H be the subgroup of M_3 generated by $\chi_{21}, \chi_{12}, \chi_{23}$. Then, $\mathcal{L}_1(M_3)$ is additively equal to the direct sum of the Lie subalgebras $\mathcal{L}_1(H)$ and $\mathcal{L}_1(\text{Inn}(F_3))$, where $\text{Inn}(F_3)$ denotes the group of inner automorphisms of F_3 . Furthermore, $\text{gr}(M_3)$ is naturally isomorphic to $\mathcal{L}_1(M_3)$ as Lie algebras if and only if $\text{gr}(H)$ is naturally isomorphic to $\mathcal{L}_1(H)$ as Lie algebras.

In [10, Theorem 1], it is shown that M_3 is a Magnus group. The proof of it was long and tedious. In Section 2, we present a rather simple proof avoiding many of the technical results. The new approach gives us the description of each quotient group $\gamma_c(M_3)/\gamma_{c+1}(M_3)$ as in Theorem 1 (1). By a result of Sjogren [14] (see, also, [5, Corollary 1.9]), M_3 satisfies the Subgroup Dimension Problem. That is, each $\gamma_c(M_3)$ is equal to the c -th dimension subgroup of M_3 . Furthermore, the new approach helps us to give a necessary and sufficient condition for a natural embedding of $\text{gr}(M_3)$ into $\mathcal{L}(\text{IA}(F_3))$. For $n \geq 2$, let $\text{Inn}(F_n)$ denote the subgroup of $\text{IA}(F_n)$ consisting of all inner automorphisms of F_n . In Section 3, by using an observation of Andreadakis [1, Section 6, p. 249], we show that $\text{gr}(\text{Inn}(F_n))$ is naturally embedded into $\mathcal{L}(\text{IA}(F_n))$. Hence, $\gamma_c(\text{Inn}(F_n))/\gamma_{c+1}(\text{Inn}(F_n))$ is isomorphic to a subgroup of $\mathcal{L}^{c+1}(\text{IA}(F_n))$ for all $c \geq 1$. Since $\text{Inn}(F_n) \cong F_n$, we obtain

$$\frac{1}{c} \sum_{d|c} \mu(d) n^{c/d} \leq \text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_n)))$$

for all n, c , with $n \geq 2$. For $c = 1$ and $n \geq 2$, we have $\text{rank}(\mathcal{L}^2(\text{IA}(F_n))) = \frac{n^2(n-1)}{2}$ (see, [1, Theorem 5.1]). For $n = 2$ we have $\text{IA}(F_2) = \text{Inn}(F_2)$, by a result of Nielsen [11] and by a result of Andreadakis [1, Theorem 6.1], we have $\text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_2))) = \frac{1}{c} \sum_{d|c} \mu(d) 2^{c/d}$. For $n = 3$, by Theorem 1(2), we may give a lower bound of $\text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_3)))$ in terms of the rank of $\mathcal{L}_1^c(H)$ for all c . In fact, we observe that

$$\frac{1}{c} \sum_{d|c} \mu(d) 2^{c/d} + \frac{1}{c} \sum_{d|c} \mu(d) 3^{c/d} \leq \text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_3)))$$

(see, Remark 2 below). For $n \geq 4$ and $c \geq 2$, Satoh [15, Corollary 3.3] provides a lower bound for $\text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_n)))$.

2 The associated Lie algebra of M_3

2.1 Lazard elimination

For a free \mathbb{Z} -module A , let $L(A)$ be the free Lie algebra on A , that is, the free Lie algebra on \mathcal{A} , where \mathcal{A} is an arbitrary \mathbb{Z} -basis of A . Thus, we may write $L(A) = L(\mathcal{A})$. For a positive integer c , let $L^c(A)$ denote the c th homogeneous component of $L(A)$. It is well-known that $L(A) = \bigoplus_{c \geq 1} L^c(A)$. For \mathbb{Z} -submodules A and B of any Lie algebra, let $[A, B]$ be the \mathbb{Z} -submodule spanned by $[a, b]$ where $a \in A$ and $b \in B$. Furthermore, $B \wr A$ denotes the \mathbb{Z} -submodule defined by $B \wr A = B + [B, A] + [B, A, A] + \cdots$.

Throughout this paper, we use the left-normed convention for Lie commutators. The following result is a version of Lazard's "Elimination Theorem" (see [2, Chapter 2, Section 2.9, Proposition 10]). In the form written here it is a special case of [3, Lemma 2.2] (see, also, [10, Section 2.2]).

Lemma 1 *Let U and V be free \mathbb{Z} -modules, and consider the free Lie algebra $L(U \oplus V)$. Then, U and $V \wr U$ freely generate Lie subalgebras $L(U)$ and $L(V \wr U)$, and there is a \mathbb{Z} -module decomposition $L(U \oplus V) = L(U) \oplus L(V \wr U)$. Furthermore, $V \wr U = V \oplus [V, U] \oplus [V, U, U] \oplus \cdots$ and, for each $n \geq 0$, there is a \mathbb{Z} -module isomorphism $[V, \underbrace{U, \dots, U}_n] \cong V \otimes \underbrace{U \otimes \cdots \otimes U}_n$ under which $[v, u_1, \dots, u_n]$ corresponds to $v \otimes u_1 \otimes \cdots \otimes u_n$ for all $v \in V$ and all $u_1, \dots, u_n \in U$.*

As a consequence of Lemma 1, we have the following result.

Corollary 1 *For free \mathbb{Z} -modules U and V , we write $L(U \oplus V)$ for the free Lie algebra on $U \oplus V$. Then, there is a \mathbb{Z} -module decomposition $L(U \oplus V) = L(U) \oplus L(V) \oplus L(W)$, where $W = W_2 \oplus W_3 \oplus \cdots$ such that, for all $m \geq 2$, W_m is the direct sum of submodules $[V, U, \underbrace{U, \dots, U}_a, \underbrace{V, \dots, V}_b]$ with $a + b = m - 2$ and $a, b \geq 0$. Each $[V, U, \underbrace{U, \dots, U}_a, \underbrace{V, \dots, V}_b]$ is isomorphic to $V \otimes U \otimes \underbrace{U \otimes \cdots \otimes U}_a \otimes \underbrace{V \otimes \cdots \otimes V}_b$ as \mathbb{Z} -module. Furthermore, $L(W)$ is the ideal of $L(U \oplus V)$ generated by the module $[V, U]$.*

2.2 A decomposition of a free Lie algebra

Let X be the free \mathbb{Z} -module with a \mathbb{Z} -basis $\{x_1, \dots, x_6\}$ and $L = L(X)$ the free Lie algebra on X . For $i = 1, 2, 3$, let $v_{2i} = x_{2i-1} + x_{2i}$. Furthermore, we write

$$U = \mathbb{Z}\text{-span}\{x_1, x_3, x_5\} \quad \text{and} \quad V = \mathbb{Z}\text{-span}\{v_2, v_4, v_6\}.$$

Since $X = U \oplus V$, we have $L = L(U \oplus V)$ and so, L is free on $\mathcal{X} = \{x_1, x_3, x_5, v_2, v_4, v_6\}$. Let \mathcal{J} be the subset of L ,

$$\begin{aligned} \mathcal{J} = & \{[v_2, x_1], [v_4, x_3], [v_6, x_5], [v_4, v_2] - [v_4, x_1], [v_2, v_4] - [v_2, x_3], [v_4, v_6] - [v_4, x_5], \\ & [v_6, x_1], [v_6, x_3], [v_2, x_5]\}. \end{aligned}$$

The aim of this section is to show the following result.

Proposition 1 *With the above notation, let $L = L(U \oplus V)$ be the free Lie algebra on $U \oplus V$. Let J be the ideal of L generated by the set \mathcal{J} . Then, $L = L(U) \oplus L(V) \oplus J$. Moreover, J is a free Lie algebra.*

For non negative integers a and b , we write $[V, U, {}_aU, {}_bV]$ for $[V, U, \underbrace{U, \dots, U}_a, \underbrace{V, \dots, V}_b]$.

By Lemma 1 and Corollary 1, we have

$$\begin{aligned} L &= L(U \oplus V) \\ &= L(U) \oplus L(V \wr U) \\ &= L(U) \oplus L(V) \oplus L(W), \end{aligned}$$

where $W = W_2 \oplus W_3 \oplus \cdots$ such that, for all $m \geq 2$,

$$W_m = \bigoplus_{a+b=m-2} [V, U, {}_aU, {}_bV].$$

Furthermore, $L(V \wr U)$ and $L(W)$ are the ideals in L generated by the modules $V \wr U$ and $[V, U]$, respectively. In particular, $L(W)$ is the ideal in L generated by the natural \mathbb{Z} -basis

$$[\mathcal{V}, \mathcal{U}] = \{[v_2, x_1], [v_4, x_3], [v_6, x_5], [v_4, x_1], [v_2, x_3], [v_4, x_5], [v_6, x_1], [v_6, x_3], [v_2, x_5]\}$$

of $[V, U]$. Let $\mathcal{X}_{V,U}$ be the natural \mathbb{Z} -basis of $V \wr U$. That is,

$$\mathcal{X}_{V,U} = \mathcal{V} \cup \left(\bigcup_{a \geq 1} [\mathcal{V}, {}_a\mathcal{U}] \right),$$

where $[\mathcal{V}, {}_a\mathcal{U}]$ is the natural \mathbb{Z} -basis of the module $[V, {}_aU]$. Let ψ_2 be the \mathbb{Z} -linear mapping from $[V, U]$ into $L(V \wr U)$ with

$$\psi_2([v_4, x_1]) = [v_4, v_2] - [v_4, x_1], \quad \psi_2([v_2, x_3]) = [v_2, v_4] - [v_2, x_3], \quad \psi_2([v_4, x_5]) = [v_4, v_6] - [v_4, x_5]$$

and ψ_2 fixes the remaining elements of $[\mathcal{V}, \mathcal{U}]$. It is clear enough that ψ_2 is a \mathbb{Z} -linear monomorphism of $[V, U]$ into $L(V \wr U)$. For $a \geq 3$, let ψ_a be the mapping from $[V, U, {}_{(a-2)}U]$ into $L(V \wr U)$ satisfying the conditions $\psi_a([v, u, u_3, \dots, u_a]) = [\psi_2([v, u]), u_3, \dots, u_a]$ for all $v \in \mathcal{V}$ and $u, u_3, \dots, u_a \in \mathcal{U}$. We define a map

$$\Psi : \mathcal{X}_{V,U} \rightarrow L(V \wr U)$$

by $\Psi(v) = v$ for all $v \in \mathcal{V}$ and, for $a \geq 2$, $\Psi(v) = \psi_a(v)$ for all $v \in [\mathcal{V}, \mathcal{U}, {}_{(a-2)}\mathcal{U}]$. Since $L(V \wr U)$ is free on $\mathcal{X}_{V,U}$, we obtain Ψ is a Lie algebra homomorphism. By applying Lemma 2.1 in [4], we see that Ψ is a Lie algebra automorphism of $L(V \wr U)$. Since $L(W)$ is a free Lie subalgebra of $L(V \wr U)$ and Ψ is an automorphism, we have $\Psi(L(W))$ is a free Lie subalgebra of $L(V \wr U)$. In fact,

$$\Psi(L(W)) = L(\Psi(W)),$$

that is, $\Psi(L(W))$ is a free Lie algebra on $\Psi(W)$.

Lemma 2 *With the above notation, $L(\Psi(W))$ is an ideal in L .*

Proof. Since Ψ is an automorphism of $L(V \wr U)$, we obtain $\Psi(L(W)) = L(\Psi(W))$ is an ideal in $L(V \wr U)$. We point out that

$$\begin{aligned} L(V \wr U) &= \Psi(L(V \wr U)) \\ (\text{By Corollary 1}) &= \Psi(L(V) \oplus L(W)) \\ (\Psi \text{ automorphism}) &= \Psi(L(V)) \oplus \Psi(L(W)) \\ &= L(V) \oplus L(\Psi(W)) \end{aligned}$$

and so,

$$L = L(U) \oplus L(V) \oplus L(\Psi(W)).$$

To show that $L(\Psi(W))$ is an ideal in L , it is enough to show that $[w, u] \in L(\Psi(W))$ for all $w \in L(\Psi(W))$ and $u \in L$. Since $L(\Psi(W))$ is an ideal in $L(V \wr U)$ and Lemma 1, it is enough to show that $[w, u] \in L(\Psi(W))$ for all $w \in L(\Psi(W))$ and $u \in L(U)$. Furthermore, we may show that $[w, x_{i_1}, \dots, x_{i_k}] \in L(\Psi(W))$ for all $w \in L(\Psi(W))$ and $x_{i_1}, \dots, x_{i_k} \in \{x_1, x_3, x_5\}$. Write

$$\mathcal{C} = \Psi([\mathcal{V}, \mathcal{U}]) \cup \left(\bigcup_{\substack{a+b \geq 1 \\ a, b \geq 0}} [\Psi([\mathcal{V}, \mathcal{U}]), {}_a\mathcal{U}, {}_b\mathcal{V}] \right). \quad (1)$$

Since \mathcal{C} is a \mathbb{Z} -basis for $\Psi(W)$, we have $L(\Psi(W)) = L(\mathcal{C})$. Thus, we need to show that $[w, x_{i_1}, \dots, x_{i_k}] \in L(\Psi(W))$ for all $w \in \mathcal{C}$ and $x_{i_1}, \dots, x_{i_k} \in \{x_1, x_3, x_5\}$. Since \mathcal{C} is a free generating set of $L(\Psi(W))$, the equation (1) and the linearity of the Lie bracket, we may assume that $w \in [\Psi([\mathcal{V}, \mathcal{U}]), {}_a\mathcal{U}, {}_b\mathcal{V}]$ with $a + b \geq 1$. Clearly, we may assume that $b \geq 1$. Since $L(\Psi(W))$ is ideal in $L(V \wr U)$ and, by the Jacobi identity, we may further assume that w has a form $[v, y_{j_1}, \dots, y_{j_a}, {}_\mu v_2, {}_\nu v_4, {}_\rho v_6]$ with $y_{j_1}, \dots, y_{j_a} \in \mathcal{U}$, $\mu, \nu, \rho \geq 0$ and $\mu + \nu + \rho \geq 1$. By using the Jacobi identity in the expression $[w, x_{i_1}, \dots, x_{i_k}]$, and replacing $[v_4, x_1]$, $[v_2, x_3]$ and $[v_4, x_5]$ by $[v_4, v_2] - \psi_2([v_4, x_1])$, $[v_2, v_4] - \psi_2([v_2, x_3])$ and $[v_4, v_6] - \psi_2([v_4, x_5])$, respectively, as many times as it is needed and since $L(\Psi(W))$ is an ideal in $L(V \wr U)$, we may show that $[w, x_{i_1}, \dots, x_{i_k}] \in L(\Psi(W))$. Therefore, $L(\Psi(W))$ is an ideal in L . \square

Example 1 In the present example, we explain the procedure described in the above proof. Let $w = [\psi_2([v_4, x_1]), x_3, v_2, v_4]$. Then,

$$\begin{aligned}
[w, x_1] &= [\psi_2([v_4, x_1]), x_3, v_2, v_4, x_1] \\
&= [\psi_2([v_4, x_1]), x_3, v_2, x_1, v_4] + [\psi_2([v_4, x_1]), x_3, v_2, [v_4, x_1]] \\
&= [\psi_2([v_4, x_1]), x_3, v_2, x_1, v_4] + \\
&\quad [\psi_2([v_4, x_1]), x_3, v_2, [v_4, v_2]] - [\psi_2([v_4, x_1]), x_3, v_2, \psi_2([v_4, x_1])] \\
&= [\psi_2([v_4, x_1]), x_3, x_1, v_2, v_4] + [\psi_2([v_4, x_1]), x_3, [v_2, x_1], v_4] + \\
&\quad [\psi_2([v_4, x_1]), x_3, v_2, [v_4, v_2]] - [\psi_2([v_4, x_1]), x_3, v_2, \psi_2([v_4, x_1])] \in L(\Psi(W)).
\end{aligned}$$

Proof of Proposition 1. Since $\psi_2([V, U]) \subseteq J$ and J is ideal in L , we get $L(\Psi(W)) \subseteq J$. Since $\mathcal{J} \subseteq L(\Psi(W))$, by Lemma 2, we obtain $J \subseteq L(\Psi(W))$. Therefore, $J = L(\Psi(W))$. That is, J is a free Lie algebra. Furthermore, $L = L(U) \oplus L(V) \oplus J$. \square

For $c \geq 2$, let $J^c = J \cap L^c$. Since J is homogeneous, we get $J = \bigoplus_{c \geq 2} J^c$. From the above proof, we have the following result.

Corollary 2 *With the above notation, let Ψ be the Lie algebra automorphism of $L(V \wr U)$ defined naturally on $V \wr U$ by means of ψ_2 . Then, $J = L(\Psi(W))$. Furthermore, for $c \geq 2$, $L^c = L^c(U) \oplus L^c(V) \oplus J^c$.*

2.3 A description of $\text{gr}(M_3)$

Our aim in this section is to show the following result. For its proof, we use similar arguments as in [10, Section 6].

Theorem 2 *Let M_3 be the McCool subgroup of the IA-automorphisms $\text{IA}(F_3)$ of F_3 . Then, $\text{gr}(M_3) \cong L/J$ as Lie algebras. In particular, $\text{gr}(M_3)$ is isomorphic as a Lie algebra to an (external) direct sum of two copies of a free Lie algebra of rank 3. Furthermore, for each c ,*

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong \gamma_c(F_3)/\gamma_{c+1}(F_3) \oplus \gamma_c(F_3)/\gamma_{c+1}(F_3)$$

as free abelian groups.

Following the notation of the previous subsection, let us denote by F the free group generated by $\{x_1, \dots, x_6\}$. It is well known that $\mathbb{L}(F)$ is a free Lie algebra of rank 6; freely generated by the set $\{x_i F' : i = 1, \dots, 6\}$. The free Lie algebras L and $\mathbb{L}(F)$ are isomorphic and from now on we identify the two Lie algebras. Furthermore, $L^c = \gamma_c(F)/\gamma_{c+1}(F)$ for all $c \geq 1$. Define

$$\begin{aligned}
r_1 &= (x_1, x_2), & r_2 &= (x_3, x_4), & r_3 &= (x_5, x_6) \\
r_4 &= (x_1 x_2, x_5), & r_5 &= (x_3 x_4, x_6), & r_6 &= (x_1 x_2, x_4) \\
r_7 &= (x_3 x_4, x_2), & r_8 &= (x_5 x_6, x_3), & r_9 &= (x_5 x_6, x_1),
\end{aligned}$$

and $\mathcal{R} = \{r_1, \dots, r_9\}$.

Let $N = \mathcal{R}^F$ be the normal closure of \mathcal{R} in F . For a positive integer d , let $N_d = N \cap \gamma_d(F)$. We point out that for $d \leq 2$, $N_d = N$. Further, for $d \geq 2$, $N_{d+1} = N_d \cap \gamma_{d+1}(F)$. Define $\mathcal{I}_d(N) = N_d \gamma_{d+1}(F)/\gamma_{d+1}(F)$. It is easily verified that $\mathcal{I}_d(N) \cong N_d/N_{d+1}$ as \mathbb{Z} -modules. The following result was shown in [10, Section 6].

Proposition 2 *For a positive integer c , N_{c+2} is generated by the set $\{(r^{\pm 1}, g_1, \dots, g_s) : r \in \mathcal{R}, s \geq c, g_1, \dots, g_s \in F \setminus \{1\}\}$. Furthermore, $\mathcal{I}_{c+2}(N) = J^{c+2}$ for all $c \geq 1$.*

Since F is residually nilpotent, we have $\bigcap_{d \geq 2} N_d = \{1\}$. Also $N \subseteq F'$, we get $\mathcal{I}_1(N) = 0$. Moreover, $\mathcal{I}_d(N) \cong N_d/N_{d+1}$ as \mathbb{Z} -modules for all $d \geq 2$, and by Proposition 2 we have, $N_d \neq N_{d+1}$ for all $d \geq 2$. Define

$$\mathcal{I}(N) = \bigoplus_{d \geq 2} N_d \gamma_{d+1}(F) / \gamma_{d+1}(F) = \bigoplus_{d \geq 2} \mathcal{I}_d(N).$$

Since N is a normal subgroup of F , we have $\mathcal{I}(N)$ is an ideal of L (see [6]).

Corollary 3 $\mathcal{I}(N) = J$.

Proof. Since $J = \bigoplus_{d \geq 2} J^d$ and $\mathcal{I}_2(N) = J^2$, we have from Proposition 2 that $\mathcal{I}(N) = J$. \square

Proof of Theorem 2. Since $M_3/M'_3 \cong F/NF' = F/F'$, we have $\mathbb{L}(M_3)$ is generated as a Lie algebra by the set $\{\alpha_i : i = 1, \dots, 6\}$ with $\alpha_i = x_i M'_3$. Since L is a free Lie algebra of rank 6 with a free generating set $\{x_1, \dots, x_6\}$, the map ζ from L into $\mathbb{L}(M_3)$ satisfying the conditions $\zeta(x_i) = \alpha_i$, $i = 1, \dots, 6$, extends uniquely to a Lie algebra homomorphism. Since $\mathbb{L}(M_3)$ is generated as a Lie algebra by the set $\{\alpha_i : i = 1, \dots, 6\}$, we have ζ is onto. Hence, $L/\text{Ker}\zeta \cong \mathbb{L}(M_3)$ as Lie algebras. By definition, $J \subseteq \text{Ker}\zeta$, and so ζ induces a Lie algebra epimorphism $\bar{\zeta}$ from L/J onto $\mathbb{L}(M_3)$. In particular, $\bar{\zeta}(x_i + J) = \alpha_i$, $i = 1, \dots, 6$. Note that $\bar{\zeta}$ induces $\bar{\zeta}_c$, say, a \mathbb{Z} -linear mapping from $(L^c + J)/J$ onto $\gamma_c(M_3)/\gamma_{c+1}(M_3)$. For $c \geq 2$,

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong \gamma_c(F)\gamma_{c+1}(F)N/\gamma_{c+1}(F)N \cong \gamma_c(F)/(\gamma_c(F) \cap \gamma_{c+1}(F)N).$$

Since $\gamma_{c+1}(F) \subseteq \gamma_c(F)$, we have by the modular law,

$$\gamma_c(F)/(\gamma_c(F) \cap \gamma_{c+1}(F)N) = \gamma_c(F)/\gamma_{c+1}(F)N_c.$$

But, by Proposition 2, for $c \geq 3$,

$$\gamma_c(F)/\gamma_{c+1}(F)N_c \cong (\gamma_c(F)/\gamma_{c+1}(F))/\mathcal{I}_c(N) \cong L^c/J^c.$$

Since $\mathcal{I}_2(N) = J^2$, we obtain, for $c \geq 2$,

$$\gamma_c(F)/\gamma_{c+1}(F)N_c \cong (\gamma_c(F)/\gamma_{c+1}(F))/\mathcal{I}_c(N) \cong L^c/J^c.$$

Therefore, for $c \geq 1$,

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong L^c/J^c \cong (L^c)^*,$$

by Corollary 2, where $(L^c)^* = L^c(U) \oplus L^c(V)$. Since both $L(U)$ and $L(V)$ are free Lie algebras of rank 3, we have $L(U) \cong L(V) \cong \text{gr}(F_3)$ in a natural way and so, for $c \geq 1$, $L^c(U) \cong L^c(V) \cong \gamma_c(F_3)/\gamma_{c+1}(F_3)$ as free abelian groups. Hence, for $c \geq 1$,

$$\gamma_c(M_3)/\gamma_{c+1}(M_3) \cong \gamma_c(F_3)/\gamma_{c+1}(F_3) \oplus \gamma_c(F_3)/\gamma_{c+1}(F_3)$$

as free abelian groups and so, $\text{rank}(\gamma_c(M_3)/\gamma_{c+1}(M_3)) = \text{rank}(L^c)^*$. Since $J = \bigoplus_{c \geq 2} J^c$, we have $(L^c + J)/J \cong L^c/(L^c \cap J) = L^c/J^c \cong (L^c)^*$ and so, we obtain $\text{Ker}\bar{\zeta}_c$ is torsion-free. Since $\text{rank}(\gamma_c(M_3)/\gamma_{c+1}(M_3)) = \text{rank}(L^c)^*$, we have $\text{Ker}\bar{\zeta}_c = \{1\}$ and so, $\bar{\zeta}_c$ is isomorphism. Since $\bar{\zeta}$ is epimorphism and each $\bar{\zeta}_c$ is isomorphism, we have $\bar{\zeta}$ is isomorphism. Hence, $L/J \cong \mathbb{L}(M_3)$ as Lie algebras. \square

3 Embeddings

In this section, we shall give a criterion for the natural embedding of $\text{gr}(M_3)$ into $\mathcal{L}_1(\text{IA}(F_3))$. We shall prove the following result.

Lemma 3 *Let H be a finitely generated subgroup of $\text{IA}(F_n)$, $n \geq 2$, with H/H' torsion-free, and let $\{y_1H', \dots, y_mH'\}$ be a \mathbb{Z} -basis for H/H' . Then, $\text{gr}(H)$ is naturally isomorphic to $\mathcal{L}_1(H)$ if and only if $\gamma_c(H) = H \cap (\text{I}_{c+1}\text{A}(F_n))$ for all c .*

Proof. We assume that $\gamma_c(H) = H \cap (\text{I}_{c+1}\text{A}(F_n))$ for all c . For $c \geq 1$, let ψ_c be the natural \mathbb{Z} -module epimorphism from $\text{gr}_c(H)$ onto $\mathcal{L}_1^c(H)$. Since $\gamma_c(H) \cap (\text{I}_{c+2}\text{A}(F_n)) = H \cap (\text{I}_{c+2}\text{A}(F_n)) = \gamma_{c+1}(H)$ for all c , we get ψ_c is isomorphism for all $c \geq 1$. Since $\text{gr}(H)$ is the (restricted) direct sum of the quotients $\text{gr}_c(H)$, there exists a group homomorphism ψ from $\text{gr}(H)$ into $\mathcal{L}_1(\text{IA}(F_n))$ such that each ψ_c is the restriction of ψ on $\text{gr}_c(H)$. It is easily shown that ψ is a Lie algebra homomorphism. Since $\psi(y_iH') = y_i(\text{I}_3\text{A}(F_n))$, $i = 1, \dots, m$, we get ψ is a Lie algebra epimorphism. Furthermore, since each ψ_c is a \mathbb{Z} -module isomorphism, we obtain ψ is a Lie algebra isomorphism. Conversely, let ϕ be a Lie algebra isomorphism from $\text{gr}(H)$ onto $\mathcal{L}_1(H)$ satisfying the conditions $\phi(y_iH') = y_i(\text{I}_3\text{A}(F_n))$, $i = 1, \dots, m$. Then, ϕ induces a \mathbb{Z} -module isomorphism ϕ_c from $\text{gr}_c(H)$ onto $\mathcal{L}_1^c(H)$ for all c . In particular, $\phi_c((y_{i_1}, \dots, y_{i_c})\gamma_{c+1}(H)) = (y_{i_1}, \dots, y_{i_c})(\text{I}_{c+2}\text{A}(F_n))$ for all $i_1, \dots, i_c \in \{1, \dots, m\}$. Furthermore, $\text{gr}_c(H)$ is \mathbb{Z} -module isomorphic to $\gamma_c(H)/(\gamma_c(H) \cap (\text{I}_{c+2}\text{A}(F_n)))$. Since $\text{gr}_c(H)$ is polycyclic and so, it is a hopfian group, we have $\gamma_{c+1}(H) = \gamma_c(H) \cap (\text{I}_{c+2}\text{A}(F_n))$ for all c . We claim that $\gamma_c(H) = H \cap (\text{I}_{c+1}\text{A}(F_n))$ for all c . Since $\gamma_c(H) \subseteq \text{I}_{c+1}\text{A}(F_n)$, it is enough to show that $H \cap (\text{I}_{c+1}\text{A}(F_n)) \subseteq \gamma_c(H)$. To get a contradiction, let $\alpha \in H \cap (\text{I}_{c+1}\text{A}(F_n))$ and $\alpha \notin \gamma_c(H)$. Since $\gamma_c(H) \subseteq \text{I}_{c+1}\text{A}(F_n)$ for all c and $\bigcap_{t \geq 2} \text{I}_t\text{A}(F_n) = \{1\}$, we get H is residually nilpotent. Thus, there exists a unique $d \in \mathbb{N}$ such that $\alpha \in \gamma_d(H) \setminus \gamma_{d+1}(H)$. Therefore, $\alpha \notin H \cap (\text{I}_{d+2}\text{A}(F_n))$. Since $\alpha \in \gamma_d(H) \setminus \gamma_{d+1}(H)$ and $\alpha \notin \gamma_c(H)$, we have $\gamma_c(H) \subseteq \gamma_{d+1}(H)$ and so, $d+1 \leq c$. Let k be a non-negative integer such that $c = d+1+k$. Since $H \cap (\text{I}_{c+1}\text{A}(F_n)) = H \cap (\text{I}_{d+2+k}\text{A}(F_n))$, we have $\alpha \in \text{I}_{d+2+k}\text{A}(F_n)$. But $\text{I}_{d+2+k}\text{A}(F_n) \subseteq \text{I}_{d+2}\text{A}(F_n)$ and so, $\alpha \in \text{I}_{d+2}\text{A}(F_n)$, which is a contradiction. Therefore, $\gamma_c(H) = H \cap (\text{I}_{c+1}\text{A}(F_n))$ for all c . \square

Remark 1 It is known that $\text{IA}(F_n)/\gamma_2(\text{IA}(F_n))$, with $n \geq 2$, is torsion-free and its rank is $\frac{n^2(n-1)}{2}$. Furthermore, $\gamma_2(\text{IA}(F_n)) = \text{I}_3\text{A}(F_n)$ (see, for example, [13]). It was conjectured by Andreadakis [1] that $\gamma_c(\text{IA}(F_n)) = \text{I}_{c+1}\text{A}(F_n)$ for all c . By Lemma 3, $\text{gr}(\text{IA}(F_n))$ is naturally isomorphic to $\mathcal{L}_1(\text{IA}(F_n))$ if and only if Andreadakis conjecture is valid. Now, if H is a finitely generated subgroup of $\text{IA}(F_n)$ with H/H' torsion free, then the statement $\text{gr}(H)$ is naturally isomorphic to $\mathcal{L}_1(H)$ seems to be an “Andreadakis Conjecture” for H .

3.1 The associated Lie algebra of $\text{Inn}(F_n)$

In this section, we show that the associated Lie algebra $\text{gr}(\text{Inn}(F_n))$ of the inner automorphisms $\text{Inn}(F_n)$ of F_n , with $n \geq 2$, is naturally embedded into $\mathcal{L}(\text{IA}(F_n))$. Throughout this section, we write $E_n = \text{Inn}(F_n)$. Recall that, for $g \in F_n$, $\tau_g(x) = gxg^{-1}$ for all $x \in F_n$. Thus, $E_n = \{\tau_g : g \in F_n\}$. The following result has been proved in [1, Section 6].

Lemma 4 *For a positive integer c , $\gamma_c(E_n) = E_n \cap \text{I}_{c+1}\text{A}(F_n)$.*

Using the above we may show the following.

Proposition 3 *Let n be positive integer, with $n \geq 2$. Then, $\text{gr}(E_n)$ is naturally embedded into $\mathcal{L}(\text{IA}(F_n))$. In particular, for all c , $\text{gr}_c(E_n)$ is isomorphic to a \mathbb{Z} -submodule of $\mathcal{L}^{c+1}(\text{IA}(F_n))$.*

Proof. Since F_n is centerless, we have $F_n \cong E_n$ in a natural way and so, E_n is finitely generated. Moreover, $\text{gr}(E_n)$ is a free Lie algebra of rank n . Namely, $\text{gr}(E_n)$ is freely generated by the set $\{\tau_{x_i} E'_n : i = 1, \dots, n\}$. Let ϕ be the mapping from $\{\tau_{x_i} E'_n : i = 1, \dots, n\}$ to $\mathcal{L}_1(E_n)$ satisfying the conditions $\phi(\tau_{x_i} E'_n) = \tau_{x_i} \text{I}_3 \text{A}(F_n)$, $i = 1, \dots, n$. Since $\text{gr}(E_n)$ is free on $\{\tau_{x_i} E'_n : i = 1, \dots, n\}$, ϕ is extended to a Lie algebra epimorphism. By Lemma 3, it is enough to show that $\gamma_c(E_n) = E_n \cap \text{I}_{c+1} \text{A}(F_n)$ for all c , which is valid by Lemma 4. Therefore, $\text{gr}(E_n)$ is naturally embedded into $\mathcal{L}(\text{IA}(F_n))$. Since

$$\begin{aligned} \mathcal{L}_1^c(E_n) &\cong \gamma_c(E_n) / (\gamma_c(E_n) \cap (\text{I}_{c+2} \text{A}(F_n))) \\ &= \gamma_c(E_n) / (E_n \cap \text{I}_{c+2} \text{A}(F_n)) \\ &= \text{gr}_c(E_n) \end{aligned}$$

for all c , we have $\text{gr}_c(E_n)$ is isomorphic to a \mathbb{Z} -submodule of $\mathcal{L}^{c+1}(\text{IA}(F_n))$ for all c . \square

Corollary 4 *For positive integers n and c , with $n \geq 2$,*

$$\frac{1}{c} \sum_{d|c} \mu(d) n^{c/d} \leq \text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_n)))$$

where μ is the Möbius function.

Proof. Since $F_n \cong E_n$ as groups and $\text{gr}(F_n)$ is a free Lie algebra, we get $\text{gr}(F_n) \cong \text{gr}(E_n)$ as Lie algebras. Hence, for $c \geq 1$, $\text{gr}_c(F_n) \cong \text{gr}_c(E_n)$ as \mathbb{Z} -modules. Since $\text{rank}(\text{gr}_c(F_n)) = \frac{1}{c} \sum_{d|c} \mu(d) n^{c/d}$, we obtain, by Proposition 3, the required result. \square

3.2 The Lie algebra $\mathcal{L}_1(M_3)$

Write $b_1 = \chi_{21}$, $b_2 = \chi_{12}$, $b_3 = \chi_{23}$, $u_2 = \chi_{31} \chi_{21}$, $u_4 = \chi_{32} \chi_{12}$ and $u_6 = \chi_{23} \chi_{13}$. Then, M_3 is generated by the set $\{b_1, b_2, b_3, u_2, u_4, u_6\}$ and its presentation is given by

$$\begin{aligned} M_3 = \langle b_1, b_2, b_3, u_2, u_4, u_6 : (u_2, b_1), (u_2, u_4 b_2^{-1}), (u_2, b_3), \\ (u_4, u_2 b_1^{-1}), (u_4, b_2), (u_4, b_3^{-1} u_6), (u_6, b_1), (u_6, b_2), (u_6, b_3) \rangle. \end{aligned}$$

We point out that $u_2 = \tau_{x_1}^{-1}$, $u_4 = \tau_{x_2}^{-1}$ and $u_6 = \tau_{x_3}^{-1}$. Let H and E be the subgroups of M_3 generated by the sets $\{b_1, b_2, b_3\}$ and $\{u_2, u_4, u_6\}$, respectively. We point out that $E = \text{Inn}(F_3)$. One can easily see that H is a free group of rank 3. Thus, $\text{gr}(H) \cong \text{gr}(E) \cong \text{gr}(F_3)$ as Lie algebras.

Proposition 4 $\mathcal{L}_1(M_3)$ is additively equal to the direct sum of the Lie subalgebras $\mathcal{L}_1(H)$ and $\mathcal{L}_1(E)$.

Proof. By the proof of Proposition 3, $\text{gr}(E) \cong \mathcal{L}_1(E)$ and so, $\mathcal{L}_1(E)$ is a free Lie algebra of rank 3. Since $b_1 \notin \text{I}_3\text{A}(F_3)$, we have $\mathcal{L}_1(H)$ is a non-trivial subalgebra of $\mathcal{L}(\text{IA}(F_3))$. In fact, $\mathcal{L}_1(H)$ is generated by the set $\{b_i(\text{I}_3\text{A}(F_3)) : i = 1, 2, 3\}$. Since $(\tau_g, \phi) = \tau_{g^{-1}\phi^{-1}(g)}$ for all $\phi \in \text{Aut}(F_3)$ and $g \in F_3$, we have $\mathcal{L}_1(E)$ is an ideal in $\mathcal{L}(\text{IA}(F_3))$ and so, $\mathcal{L}_1(H) + \mathcal{L}_1(E)$ is a Lie subalgebra of $\mathcal{L}_1(M_3)$. Let $\bar{w} \in \mathcal{L}_1(H) \cap \mathcal{L}_1(E)$. Since both $\mathcal{L}_1(H)$ and $\mathcal{L}_1(E)$ are graded Lie algebras, we may assume that $\bar{w} \in \mathcal{L}_1^d(H) \cap \mathcal{L}_1^d(E)$ for some d . Thus, there are $u \in \gamma_d(H)$ and $v \in \gamma_d(E)$ such that $\bar{w} = u(\text{I}_{d+2}\text{A}(F_3)) = v(\text{I}_{d+2}\text{A}(F_3))$. To get a contradiction, we assume that $u, v \notin \text{I}_{d+2}\text{A}(F_3)$. Therefore, $v \in \gamma_d(E) \setminus \gamma_{d+1}(E)$ and so, there exists $\omega \in \gamma_d(F_3) \setminus \gamma_{d+1}(F_3)$ such that $v = \tau_\omega \rho$, where $\rho \in \gamma_{d+1}(E)$. Since $\gamma_{d+1}(E) \subseteq \text{I}_{d+2}\text{A}(F_3)$, we get $u^{-1}\tau_\omega \in \text{I}_{d+2}\text{A}(F_3)$. Since u^{-1} fixes x_3 , we have $x_3^{-1}(u^{-1}\tau_\omega(x_3)) = (x_3, u^{-1}(\omega^{-1})) \in \gamma_{d+2}(F_3)$. Since $u^{-1}(x_j) = x_j y_j$, $y_j \in \gamma_{d+1}(F_3)$, $j = 1, 2$, and $\gamma_d(F_3)$ is a fully invariant subgroup of F_3 , we have $u^{-1}(\omega^{-1}) = \omega^{-1}\omega_1$, with $\omega_1 \in \gamma_{d+1}(F_3)$ and so, $(w, x_3) \in \gamma_{d+2}(F_3)$. Since $\text{gr}(F_3)$ is a free Lie algebra of rank 3 with a free generating set $\{x_i F_3' : i = 1, 2, 3\}$ and $\gamma_d(F_3)/\gamma_{d+1}(F_3)$ is the d -th homogeneous component of $\text{gr}(F_3)$, we obtain $(w, x_3) \in \gamma_{d+1}(F_3) \setminus \gamma_{d+2}(F_3)$, which is a contradiction. Therefore, $\mathcal{L}_1(H) \cap \mathcal{L}_1(E) = \{0\}$. By the proof of Theorem 2, we obtain $\mathcal{L}_1(M_3) = \mathcal{L}_1(H) \oplus \mathcal{L}_1(E)$. \square

Remark 2 By Proposition 4, for all c , we obtain $\mathcal{L}_1^c(M_3) = \mathcal{L}_1^c(H) \oplus \mathcal{L}_1^c(E)$. Since $\mathcal{L}_1^c(E) \cong \gamma_c(E)/\gamma_{c+1}(E) \cong \gamma_c(F_3)/\gamma_{c+1}(F_3)$, we have $\text{rank}(\mathcal{L}_1^c(E)) = \frac{1}{c} \sum_{d|c} \mu(d)3^{c/d}$. Thus, for any c ,

$$\text{rank}(\mathcal{L}_1^c(H)) + \frac{1}{c} \sum_{d|c} \mu(d)3^{c/d} \leq \text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_3))).$$

Let H_1 be the subgroup of H generated by the set $\{\chi_{21}, \chi_{23}\}$. We point out that H_1 is a free group of rank 2. Then, $\text{gr}(H_1)$ is a free Lie algebra of rank 2. Since both χ_{21} and χ_{23} fix x_1 and x_3 , it may be shown that $\gamma_c(H_1) = H_1 \cap \text{I}_{c+1}\text{A}(F_3)$ for all c . Therefore, $\gamma_c(H_1) \cap \text{I}_{c+2}\text{A}(F_3) = \gamma_{c+1}(H_1)$ for all c and so, $\mathcal{L}_1^c(H_1) \cong \text{gr}_c(H_1)$ for all $c \geq 1$. Since $\text{gr}(H_1)$ is a free Lie algebra on $\mathcal{H}_1 = \{\chi_{21}H_1', \chi_{23}H_1'\}$, the mapping ψ from \mathcal{H}_1 into $\mathcal{L}_1(H_1)$ satisfying the conditions $\psi(aH_1') = a(\text{I}_3\text{A}(F_3))$ with $a \in \{\chi_{21}, \chi_{23}\}$ can be extended to a Lie algebra homomorphism $\bar{\psi}$. Since $\mathcal{L}_1(H_1)$ is generated as Lie algebra by the set $\{\chi_{21}(\text{I}_3\text{A}(F_3)), \chi_{23}(\text{I}_3\text{A}(F_3))\}$, we have $\bar{\psi}$ is onto. By Lemma 3, $\bar{\psi}$ is a natural Lie algebra isomorphism from $\text{gr}(H_1)$ onto $\mathcal{L}_1(H_1)$. Since $\mathcal{L}_1^c(H_1) \cong \text{gr}_c(H_1)$ for all $c \geq 1$, we have $\text{rank}(\mathcal{L}_1^c(H_1)) = \frac{1}{c} \sum_{d|c} \mu(d)2^{c/d}$. Since $\mathcal{L}_1(H_1)$ is a Lie subalgebra of $\mathcal{L}_1(H)$, we obtain $\mathcal{L}_1^c(H_1) \leq \mathcal{L}_1^c(H)$ for all c . Therefore,

$$\frac{1}{c} \sum_{d|c} \mu(d)2^{c/d} \leq \text{rank}(\mathcal{L}_1^c(H))$$

for all c and so,

$$\frac{1}{c} \sum_{d|c} \mu(d)2^{c/d} + \frac{1}{c} \sum_{d|c} \mu(d)3^{c/d} \leq \text{rank}(\mathcal{L}^{c+1}(\text{IA}(F_3)))$$

for all c .

In our next result, we give a necessary and sufficient condition for a natural embedding of $\text{gr}(M_3)$ into $\mathcal{L}(\text{IA}(F_3))$.

Proposition 5 *Let H be the subgroup of M_3 generated by $\chi_{21}, \chi_{12}, \chi_{23}$. Then, $\text{gr}(M_3)$ is naturally isomorphic to $\mathcal{L}_1(M_3)$ as Lie algebras if and only if $\text{gr}(H)$ is naturally isomorphic to $\mathcal{L}_1(H)$ as Lie algebras.*

Proof. Suppose that $\text{gr}(M_3)$ is naturally isomorphic to $\mathcal{L}_1(M_3)$ as Lie algebras. Thus, for all c , $\text{gr}_c(M_3) \cong \mathcal{L}_1^c(M_3)$ as \mathbb{Z} -modules. By Proposition 4, $\text{gr}_c(M_3) \cong \mathcal{L}_1^c(H) \oplus \mathcal{L}_1^c(E)$. Since, for all c , $\text{gr}_c(M_3) \cong \text{gr}_c(F_3) \oplus \text{gr}_c(F_3)$ as \mathbb{Z} -modules, we obtain

$$\begin{aligned} \text{rank}(\mathcal{L}_1^c(H)) &= \text{rank}(\mathcal{L}_1^c(E)) \\ &= \text{rank}(\text{gr}_c(F_3)) \\ &= \text{rank}(\text{gr}_c(H)). \end{aligned}$$

Therefore, $\mathcal{L}_1^c(H) \cong \text{gr}_c(H)$ for all c . Hence, $\gamma_c(H) \cap \text{I}_{c+2}\text{A}(F_3) = \gamma_{c+1}(H)$ for all c and so, as in the proof of Lemma 3, $\gamma_c(H) = H \cap \text{I}_{c+1}\text{A}(F_3)$ for all c . We point out that H/H' is torsion-free with a free generating set $\mathcal{H} = \{\chi_{21}H', \chi_{12}H', \chi_{23}H'\}$. Since $\text{gr}(H)$ is a free Lie algebra with a free generating set \mathcal{H} , and $\mathcal{L}_1(H)$ is generated as a Lie algebra by the set $\{\chi_{21}(\text{I}_3\text{A}(F_3)), \chi_{12}(\text{I}_3\text{A}(F_3)), \chi_{23}(\text{I}_3\text{A}(F_3))\}$, we have there exists a natural Lie epimorphism from $\text{gr}(H)$ onto $\mathcal{L}_1(H)$. By Lemma 3, $\text{gr}(H)$ is naturally isomorphic to $\mathcal{L}_1(H)$ as Lie algebras.

Conversely, let $\text{gr}(H)$ be naturally isomorphic to $\mathcal{L}_1(H)$ as Lie algebras. By Lemma 3, $\gamma_c(H) = H \cap (\text{I}_{c+1}\text{A}(F_3))$ for all c . Thus,

$$\begin{aligned} \gamma_c(H) \cap \text{I}_{c+2}\text{A}(F_3) &= H \cap \text{I}_{c+1}\text{A}(F_3) \cap \text{I}_{c+2}\text{A}(F_3) \\ &= H \cap \text{I}_{c+2}\text{A}(F_3) \\ &= \gamma_{c+1}(H) \end{aligned}$$

and so, $\mathcal{L}_1^c(H) \cong \text{gr}_c(H)$ for all c . By Proposition 4,

$$\mathcal{L}_1^c(M_3) = \mathcal{L}_1^c(H) \oplus \mathcal{L}_1^c(E)$$

for all c , and by Theorem 2,

$$\mathcal{L}_1^c(M_3) \cong \text{gr}_c(M_3)$$

for all c . Therefore,

$$\gamma_c(M_3) \cap \text{I}_{c+2}\text{A}(F_3) = \gamma_{c+1}(M_3)$$

for all c . As in the proof of Lemma 3, for all c ,

$$M_3 \cap \text{I}_{c+1}\text{A}(F_3) = \gamma_c(M_3).$$

Let ψ be the Lie algebra epimorphism from L onto $\mathcal{L}_1(M_3)$ satisfying the conditions $\psi(x_{2j-1}) = b_j(\text{I}_3\text{A}(F_3))$ and $\psi(v_{2j}) = \tau_{x_{2j-1}}^{-1}(\text{I}_3\text{A}(F_3))$, $j = 1, 2, 3$. By τ_w we denote the inner automorphisms of F_3 with $\tau_w(x) = w^{-1}xw$. Since $J \subseteq \text{Ker}\psi$ and $\text{gr}(M_3) \cong L/J$ as Lie algebras, there exists a Lie algebra epimorphism $\bar{\psi}$ from $\text{gr}(M_3)$ onto $\mathcal{L}_1(M_3)$ satisfying the conditions $\bar{\psi}(b_i M'_3) = b_i(\text{I}_3\text{A}(F_3))$ and $\bar{\psi}(\tau_{x_i}^{-1} M'_3) = \tau_{x_i}^{-1}(\text{I}_3\text{A}(F_3))$, $i = 1, 2, 3$. Since, in addition, $M_3 \cap \text{I}_{c+1}\text{A}(F_3) = \gamma_c(M_3)$ for all c , we obtain from Lemma 3 the required result. \square

Remark 3 For $c = 1, 2, 3$, we show that $\gamma_c(H) = H \cap \text{I}_{c+1}\text{A}(F_3)$. That is, $\mathcal{L}_1^c(H) \cong \gamma_c(H)/\gamma_{c+1}(H)$ for $c = 1, 2, 3$. For $c \geq 4$, the method used becomes very cumbersome. For simplicity, let $x = \chi_{21}$, $y = \chi_{12}$ and $z = \chi_{23}$. For $c = 1$, our claim is trivially true. Let $c = 2$. Since $\gamma_2(H)/\gamma_3(H)$ is a free abelian group of rank 3, we have each element $h \in \gamma_2(H) \setminus \gamma_3(H)$ is uniquely written as

$$h = (z, x)^{a_1} (z, y)^{a_2} (y, x)^{a_3} v,$$

where $v \in \gamma_3(H)$ and a_1, a_2, a_3 are non-negative integers. We claim that $h \notin \text{I}_4\text{A}(F_3)$. By a direct calculation, $(z, x)^{a_1}(x_i) = x_i$, $i \neq 2$, and $(z, x)^{a_1}(x_2) = x_2(x_3, x_1, x_2)^{-a_1}u$, where $u \in \gamma_4(F_3)$. Working modulo $\gamma_3(H)$, we have

$$(z, \tau_{x_2^{-1}})^{a_2} \equiv (z, y)^{a_2}(z, \chi_{32})^{a_2} \quad \text{and} \quad (\tau_{x_2^{-1}}, x)^{a_3} \equiv (y, x)^{a_3}(\chi_{32}, x)^{a_3}.$$

We point out that, for $\phi \in \text{IA}(F_3)$ and $g \in F_3$, we have $(\tau_g, \phi) = \tau_{g^{-1}\phi^{-1}(g)}$. Since $(z, \tau_{x_2^{-1}}) = \tau_{(x_3^{-1}, x_2^{-1})}$ and $(\tau_{x_2^{-1}}, x) = \tau_{(x_2^{-1}, x_1^{-1})}$, we get

$$(\tau_{(x_3^{-1}, x_2^{-1})})^{a_2} \equiv (z, y)^{a_2}(z, \chi_{32})^{a_2} \quad \text{and} \quad (\tau_{(x_2^{-1}, x_1^{-1})})^{a_3} \equiv (y, x)^{a_3}(\chi_{32}, x)^{a_3}. \quad (*)$$

Since both $(z, \chi_{32})^{a_2}$ and $(\chi_{32}, x)^{a_3}$ fix x_1 , we obtain

$$(z, y)^{a_2}(x_1) = x_1(x_1, (x_3, x_2))^{a_2}v_1 \quad \text{and} \quad (y, x)^{a_3}(x_1) = x_1(x_1, (x_2, x_1))^{a_3}v_2,$$

where $v_1, v_2 \in \gamma_4(F_3)$. Thus, $(z, y), (y, x) \in \text{I}_3\text{A}(F_3) \setminus \text{I}_4\text{A}(F_3)$. Furthermore, it is easily shown that, for $a_1 + a_2 + a_3 \neq 0$, $h \in \gamma_2(H)$ and $h \notin \text{I}_4\text{A}(F_3)$. Therefore, $\gamma_2(H) = H \cap \text{I}_3\text{A}(F_3)$.

For $c = 3$, we apply similar arguments as before. We point out that $\gamma_3(H)/\gamma_4(H)$ is a free abelian group of rank 8. Each element $g \in \gamma_3(H) \setminus \gamma_4(H)$ is uniquely written as $g = g_1 g_2 g_3 u$, where $g_1 = (z, x, x)^{b_1}(z, x, z)^{b_2}$, $g_2 = (y, x, x)^{b_3}(y, x, z)^{b_4}(z, x, y)^{b_5}(z, y, z)^{b_6}$, $g_3 = (y, x, y)^{b_7}(z, y, y)^{b_8}$, $u \in \gamma_4(H)$ and b_1, \dots, b_8 are non-negative integers. Since $g_1(x_i) = x_i$ for $i \neq 1, 3$, we have, by direct calculations,

$$g_1(x_2) = x_2(x_3, x_1, x_1, x_2)^{-b_1}(x_3, x_1, x_2, x_3)^{-b_2}(x_3, x_2, (x_3, x_1))^{b_2}v_{12}, \quad (1)$$

where $v_{12} \in \gamma_5(F_3)$. By the equation (*), and working modulo $\gamma_4(H)$, we have

$$\tau_{(x_2, x_1, x_1)}^{b_3} \equiv (y, x, x)^{b_3}(\chi_{32}, x, x)^{b_3} \quad \text{and} \quad \tau_{(x_3, x_2, x_1)}^{b_4} \equiv (y, x, z)^{b_4}(\chi_{32}, x, z)^{b_4}.$$

Since both $(\chi_{32}, x, x)^{b_3}$ and $(\chi_{32}, x, z)^{b_4}$ fix x_1 , we get

$$(y, x, x)^{b_3}(x_1) = x_1(x_1, (x_2, x_1, x_1))^{b_3}v_3 \quad \text{and} \quad (y, x, z)^{b_4}(x_1) = x_1(x_1, (x_3, x_2, x_1))^{b_4}v_4,$$

where $v_3, v_4 \in \gamma_5(F_3)$. Since $(z, x)(x_2) = x_2(x_3, x_1, x_2)^{-1}u$, with $u \in \gamma_4(F_3)$, and by the equation (*), we get

$$(z, x, \tau_{x_2^{-1}})^{b_5} \equiv \tau_{(x_3, x_1, x_2)^{-1}}^{b_5} \equiv (z, x, y)^{b_5}(z, x, \chi_{32})^{b_5}$$

and

$$(\tau_{(x_3^{-1}, x_2^{-1})}, \chi_{23})^{b_6} \equiv \tau_{(x_3, x_2, x_3^{-1})}^{b_6} \equiv (z, y, z)^{b_6}(z, y, \chi_{32})^{b_6}.$$

Since both $(z, x, \chi_{32})^{b_5}$ and $(z, y, \chi_{32})^{b_6}$ fix x_1 , we obtain

$$(z, x, y)^{b_5}(x_1) = x_1(x_3, x_1, x_2, x_1)^{b_5}v_5 \quad \text{and} \quad (z, y, z)^{b_6}(x_1) = x_1(x_3, x_2, x_3, x_1)^{b_6}v_6,$$

where $v_5, v_6 \in \gamma_5(F_3)$. Therefore,

$$g_2(x_1) = x_1(x_2, x_1, x_1, x_1)^{-b_3}(x_3, x_2, x_1, x_1)^{-b_4}(x_3, x_1, x_2, x_1)^{b_5}(x_3, x_2, x_3, x_1)^{b_6}v, \quad (2)$$

where $v \in \gamma_5(F_3)$.

Next, we point out that $(\chi_{32}, \chi_{21})(x_i) = x_i$ for $i = 1, 2$ and $(\chi_{32}, \chi_{21})(x_3) = x_3(x_2, x_1, x_3)w$, where $w \in \gamma_4(F_3)$. Since $(\chi_{32}, \chi_{21}, \tau_{x_2}^{-1}) = \text{Id}_{F_3}$, we have

$$(\chi_{32}, x, y) = (\chi_{32}, x, \chi_{32})^{-1}\psi,$$

where $\psi \in \gamma_4(M_3)$. Hence,

$$(\tau_{(\tau_2^{-1}, \tau_1^{-1})}, y)^{b_7} \equiv (y, x, y)^{b_7} (\chi_{32}, x, \chi_{32})^{-b_7}$$

and so,

$$(\tau_{(x_2^{-1}, x_1^{-1})}, \tau_{x_2^{-1}} \chi_{32}^{-1})^{b_7} \equiv (y, x, y)^{b_7} (\chi_{32}, x, \chi_{32})^{-b_7}.$$

Since $(\tau_{(x_2^{-1}, x_1^{-1})}, \chi_{32}^{-1})^{b_7} = \text{Id}_{F_3}$, we get

$$(\tau_{(x_2^{-1}, x_1^{-1})}, \tau_{x_2^{-1}})^{b_7} \equiv (y, x, y)^{b_7} (\chi_{32}, x, \chi_{32})^{-b_7}.$$

Therefore,

$$(y, x, y)^{b_7}(x_1) = x_1(x_2, x_1, x_1, x_2)^{b_7} v_7,$$

where $v_7 \in \gamma_5(F_3)$. By direct calculations, $(z, \chi_{32})(x_1) = x_1$, $(z, \chi_{32})(x_2) = x_2(x_3, x_2, x_2)^{-1}u_2$ and $(z, \chi_{32})(x_3) = x_3(x_3, x_2, x_3)u_3$, where $u_2, u_3 \in \gamma_4(F_3)$. It is easily verified that

$$(z, \chi_{32}, \tau_{x_2}^{-1}) = \tau_{(x_3, x_2, x_2)^{-1}} \psi_1,$$

where $\psi_1 \in \gamma_4(M_3)$. By the equation (*),

$$(\tau_{(x_3^{-1}, x_2^{-1})}, y)^{b_8} \equiv (z, y, y)^{b_8} (z, \chi_{32}, y)^{b_8}.$$

Since

$$(\tau_{(x_3^{-1}, x_2^{-1})}, y)^{b_8} \equiv (\tau_{(x_3^{-1}, x_2^{-1})}, \tau_{x_2^{-1}})^{b_8} (\tau_{(x_3^{-1}, x_2^{-1})}, \chi_{32}^{-1})^{b_8}$$

and

$$(z, \chi_{32}, y)^{b_8} \equiv (z, \chi_{32}, \tau_{x_2}^{-1})^{b_8} (z, \chi_{32}, \chi_{32})^{-b_8},$$

we have

$$(\tau_{(x_3, x_2)}, \chi_{32}^{-1})^{b_8} \equiv (z, y, y)^{b_8} (z, \chi_{32}, \chi_{32})^{-b_8}.$$

Hence,

$$\tau_{(x_3, x_2, x_2)^{-1}}^{b_8} \equiv (z, y, y)^{b_8} (z, \chi_{32}, \chi_{32})^{-b_8}.$$

Since $(z, \chi_{32}, \chi_{32})$ fixes x_1 , we have

$$(z, y, y)^{b_8}(x_1) = x_1(x_3, x_2, x_2, x_1)^{b_8} v_8,$$

where $v_8 \in \gamma_5(F_3)$. Therefore,

$$g_3(x_1) = x_1(x_2, x_1, x_1, x_2)^{b_7} (x_3, x_2, x_2, x_1)^{b_8} v_{7,8}, \quad (3)$$

where $v_{7,8} \in \gamma_5(F_3)$. Since the "basic" group commutators of length 4 consists of a basis for $\gamma_4(F_3)/\gamma_5(F_3)$, we obtain from the equations (1), (2) and (3) that $g \in \gamma_4(H)$ and $g \notin \text{I}_6\text{A}(F_3)$, that is, $\gamma_4(H) = H \cap \text{I}_5\text{A}(F_3)$.

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